高等数学A

第3章 一元函数积分学

3.2 定积分

3.2.5 用换元法计算定积分
3.2.6 用分部积分法计算定积分

中南大学开放式精品示范课堂高等数学建设组

3.2 定积分

定积分的计算

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一、定积分的换元法

换元法的引入　考虑计算$\int_0^a \sqrt{a^2-x^2}dx$,

由Newton-Leibniz公式可分三步求得:

$$\int \sqrt{a^2-x^2}dx \overset{x=a\sin t}{=\!=\!=\!=} \int a^2\cos^2 t dt = \frac{a^2}{2}t + \frac{a^2}{4}\sin 2t + C$$

$$\overset{t=\arcsin\frac{x}{a}}{=\!=\!=\!=\!=\!=} \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$$

$$\int_0^a \sqrt{a^2-x^2}dx = \left[\frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2}\right]_0^a = \frac{\pi a^2}{4}$$

$$\int_0^a \sqrt{a^2-x^2}dx \overset{x=a\sin t}{=\!=\!=\!=} \int_0^{\frac{\pi}{2}} a^2\cos^2 t dt = \left(\frac{a^2}{2}t + \frac{a^2}{4}\sin 2t\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi a^2}{4}$$

定理 假设(1) $f(x)$在$[a,b]$上连续；

(2) 函数$x=\varphi(t)$在$[\alpha,\beta]$上是单值的且有连续导数；

(3) 当t在区间$[\alpha,\beta]$上变化时，$x=\varphi(t)$的值在$[a,b]$上变化，且$\varphi(\alpha)=a$、$\varphi(\beta)=b$，

则有$\int_a^b f(x)dx=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$.

证 设$F(x)$是$f(x)$的一个原函数，即$F'(x)=f(x)$，

则 $\int_a^b f(x)dx=F(b)-F(a)$,

而$F(x)$与$x=\varphi(t)$可复合成函数$\Phi(t)=F[\varphi(t)]$,

且$\Phi'(t)=\dfrac{dF}{dx}\cdot\dfrac{dx}{dt}=f(x)\varphi'(t)=f[\varphi(t)]\varphi'(t)$,

$\therefore\ \Phi(t)$是$f[\varphi(t)]\varphi'(t)$的一个原函数.

$$\int_{\alpha}^{\beta} f[\varphi(t)]\varphi'(t)dt=\Phi(\beta)-\Phi(\alpha)=F[\varphi(\beta)]-F[\varphi(\alpha)]$$

$$=F(b)-F(a),$$

故结论成立.

注意:

(1)当$\alpha > \beta$时，换元公式仍成立.

(2)$x=\varphi(t)$在$[a,b]$上单值,从而使得每一个$x\in[a,b]$都有唯一确定的$t\in[\alpha,\beta]$与它相对应

(3) $a<b$时,未必$\alpha<\beta$.

(4)在应用换元积分法时换元同时换限.

(5)此换元法对应于不定积分中的第二换元法；也可倒过来使用，这样就对应于第一换元法，即

$$\int_{\alpha}^{\beta} f[\varphi(t)]\varphi'(t)dt = \int_{a}^{b} f(x)dx$$

令$\varphi(t)=x$,且$\varphi(\alpha)=a,\varphi(\beta)=b$

应用换元公式时应注意:

（1）用 $x=\varphi(t)$ 把变量 x 换成新变量 t 时，积分限也相应的改变.

（2）求出 $f[\varphi(t)]\varphi'(t)$ 的一个原函数 $\Phi(t)$ 后，不必象计算不定积分那样再要把 $\Phi(t)$ 变换成原变量 x 的函数，而只要把新变量 t 的上、下限分别代入 $\Phi(t)$ 然后相减就行了.

定积分换元法习例

例1 计算 $\int_0^{\frac{\pi}{2}} \cos^5 x \sin x dx.$ **例2** 计算 $\int_0^{\pi} \sqrt{\sin^3 x - \sin^5 x} dx.$

例3 计算 $\int_{\sqrt{e}}^{e^{\frac{3}{4}}} \frac{dx}{x\sqrt{\ln x(1-\ln x)}}.$ **例4** 计算 $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx.$

例5 计算 $\int_0^4 \frac{dx}{1+\sqrt{x}}.$ **例6** 计算 $\int_0^a \frac{1}{x+\sqrt{a^2-x^2}} dx. \quad (a>0)$

例7 证明:(1)若$f(x)$在$[-a, a]$上连续且为偶函数,则

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

(2)若$f(x)$在$[-a, a]$上连续且为奇函数，则$\int_{-a}^{a} f(x)dx = 0$

(3)若$\varphi(u)$连续,则$\int_{-a}^{a} \varphi(x^2)dx = 2\int_0^a \varphi(x^2)dx$

例8 计算 $\int_{-1}^{1} \frac{2x^2 + x\cos x}{1+\sqrt{1-x^2}} dx.$

例9 证明 (1) $\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx.$

(2) $\int_0^{\pi} xf(\sin x)dx = \pi\int_0^{\frac{\pi}{2}} f(\sin x)dx.$

(3) $\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx.$

由此计算 $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$, 其中 $f(x)$ 在 $[0,1]$ 上连续.

例10 设 $f(x)=\begin{cases} \frac{1}{1+x} & x\geq 0 \\ \frac{1}{1+e^x} & x<0 \end{cases}$ 求 $\int_0^2 f(x-1)dx.$

例11 设 $f(x)$ 是以 T 为周期的连续函数, 证明 $\int_a^{a+T} f(x)dx$ 的值与 a 无关.

例1 计算 $\int_{0}^{\frac{\pi}{2}} \cos^{5} x \sin x dx.$

解 令 $t = \cos x$, $dt = d\cos x = -\sin x dx$,

$$x = \frac{\pi}{2} \Rightarrow t = 0, \qquad x = 0 \Rightarrow t = 1,$$

$$\int_{0}^{\frac{\pi}{2}} \cos^{5} x \sin x dx = -\int_{0}^{\frac{\pi}{2}} \cos^{5} x d\cos x$$

$$= -\int_{1}^{0} t^{5} dt = \int_{0}^{1} t^{5} dt$$

$$= \left.\frac{t^{6}}{6}\right|_{0}^{1} = \frac{1}{6}.$$

例2 计算 $\int_0^{\pi}\sqrt{\sin^3 x-\sin^5 x}\,dx.$

解 $\because f(x)=\sqrt{\sin^3 x-\sin^5 x}=|\cos x|(\sin x)^{\frac{3}{2}}$

$$\therefore \int_0^{\pi}\sqrt{\sin^3 x-\sin^5 x}\,dx=\int_0^{\pi}|\cos x|(\sin x)^{\frac{3}{2}}dx$$

$$=\int_0^{\frac{\pi}{2}}\cos x(\sin x)^{\frac{3}{2}}dx-\int_{\frac{\pi}{2}}^{\pi}\cos x(\sin x)^{\frac{3}{2}}dx$$

$$=\int_0^{\frac{\pi}{2}}(\sin x)^{\frac{3}{2}}d\sin x-\int_{\frac{\pi}{2}}^{\pi}(\sin x)^{\frac{3}{2}}d\sin x$$

$$=\frac{2}{5}(\sin x)^{\frac{5}{2}}\Big|_0^{\frac{\pi}{2}}-\frac{2}{5}(\sin x)^{\frac{5}{2}}\Big|_{\frac{\pi}{2}}^{\pi}=\frac{4}{5}.$$

例3 计算 $\int_{\sqrt{e}}^{e^{\frac{3}{4}}} \frac{dx}{x\sqrt{\ln x(1-\ln x)}}$.

解 原式 $=\int_{\sqrt{e}}^{e^{\frac{3}{4}}} \frac{d(\ln x)}{\sqrt{\ln x(1-\ln x)}}$

$$=\int_{\sqrt{e}}^{e^{\frac{3}{4}}} \frac{d(\ln x)}{\sqrt{\ln x}\sqrt{(1-\ln x)}} = 2\int_{\sqrt{e}}^{e^{\frac{3}{4}}} \frac{d\sqrt{\ln x}}{\sqrt{1-(\sqrt{\ln x})^2}}$$

$$=2\left[\arcsin(\sqrt{\ln x})\right]_{\sqrt{e}}^{e^{\frac{3}{4}}} = \frac{\pi}{6}.$$

例4 计算$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx.$

解 原式$\xlongequal{x=\frac{\pi}{2}-t} \int_{\frac{\pi}{2}}^{0} \frac{\sin t}{\sin t + \cos t}(-dt)$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin t}{\sin t + \cos t} dt$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin t + \cos t}{\sin t + \cos t} dt = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

例5 计算$\int_0^4 \frac{dx}{1+\sqrt{x}}$.

解 $$\int_0^4 \frac{dx}{1+\sqrt{x}} \overset{\sqrt{x}=t}{=\!=\!=} \int_0^2 \frac{2t}{1+t}dt$$

$$= 2\int_0^2 (1 - \frac{1}{1+t})dt$$

$$= 2(t - \ln(1+t))\Big|_0^2 = 2(2 - \ln 3)$$

例6 计算 $\int_0^a \frac{1}{x+\sqrt{a^2-x^2}}dx.\quad (a>0)$

解 令 $x=a\sin t,\quad dx=a\cos t dt,$

$$x=a \Rightarrow t=\frac{\pi}{2},\quad x=0 \Rightarrow t=0,$$

$$\text{原式}=\int_0^{\frac{\pi}{2}} \frac{a\cos t}{a\sin t+\sqrt{a^2(1-\sin^2 t)}}dt$$

$$=\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t+\cos t}dt = \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(1+\frac{\cos t-\sin t}{\sin t+\cos t}\right)dt$$

$$=\frac{1}{2}\cdot\frac{\pi}{2}+\frac{1}{2}\left[\ln\left|\sin t+\cos t\right|\right]_0^{\frac{\pi}{2}}=\frac{\pi}{4}.$$

例7 证明:(1)若$f(x)$在$[-a, a]$上连续且为偶函数,则

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

(2)若$f(x)$在$[-a, a]$上连续且为奇函数，则 $\int_{-a}^{a} f(x)dx = 0$

(3)若$\varphi(u)$连续,则$\int_{-a}^{a}\varphi(x^2)dx = 2\int_{0}^{a}\varphi(x^2)dx$

证

$$(1)\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$\overset{x=-t}{=\!=} \int_{a}^{0} f(-t)(-dt) + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

$$(2)\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= \int_{a}^{0} f(-t)(-dt) + \int_{0}^{a} f(x)dx = 0$$

(3)$\varphi(x^2)$为$[-a, a]$上连续的偶函数,由(1)结论成立.

例8 计算 $\int_{-1}^{1}\frac{2x^2+x\cos x}{1+\sqrt{1-x^2}}dx.$

解 原式 $=\int_{-1}^{1}\frac{2x^2}{1+\sqrt{1-x^2}}dx+\int_{-1}^{1}\frac{x\cos x}{1+\sqrt{1-x^2}}dx$

偶函数　　奇函数

$$=4\int_{0}^{1}\frac{x^2}{1+\sqrt{1-x^2}}dx=4\int_{0}^{1}\frac{x^2(1-\sqrt{1-x^2})}{1-(1-x^2)}dx$$

$$=4\int_{0}^{1}(1-\sqrt{1-x^2})dx=4-4\int_{0}^{1}\sqrt{1-x^2}dx$$

单位圆的面积

$$=4-\pi.$$

例9 证明 (1) $\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx.$

(2) $\int_0^{\pi} xf(\sin x)dx = \pi\int_0^{\frac{\pi}{2}} f(\sin x)dx.$

(3) $\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx.$

由此计算 $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$，其中 $f(x)$ 在 $[0,1]$ 上连续.

证 (1) 令 $x = \frac{\pi}{2} - t, \quad \Rightarrow dx = -dt,$

$$\int_0^{\frac{\pi}{2}} f(\sin x)dx = -\int_{\frac{\pi}{2}}^{0} f\left[\sin\left(\frac{\pi}{2} - t\right)\right]dt$$

$$= \int_0^{\frac{\pi}{2}} f(\cos t)dt = \int_0^{\frac{\pi}{2}} f(\cos x)dx;$$

(2) 令$x=\frac{\pi}{2}-t$, $\Rightarrow dx=-dt$,

$$\int_0^{\pi} xf(\sin x)dx=-\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}(\frac{\pi}{2}-t)\cdot f\left[\sin\left(\frac{\pi}{2}-t\right)\right]dt$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\frac{\pi}{2}-t)\cdot f(\cos t)dt$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\pi}{2}f(\cos t)dt-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}tf(\cos t)dt$$

$$=2\int_0^{\frac{\pi}{2}}\frac{\pi}{2}f(\cos t)dt-0$$

$$=\pi\int_0^{\frac{\pi}{2}}f(\cos t)dt=\pi\int_0^{\frac{\pi}{2}}f(\sin x)dx.$$

(3) 令$x=\pi-t, dx=-dt$

$$\int_0^{\pi} x f(\sin x)dx=\int_{\pi}^{0}(\pi-t)f(\sin t)(-dt)$$

$$=\int_0^{\pi}(\pi-t)f(\sin t)dt$$

$$=\pi\int_0^{\pi} f(\sin t)dt-\int_0^{\pi} tf(\sin t)dt$$

$$\therefore \int_0^{\pi} xf(\sin x)dx=\frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$$

从而$\displaystyle\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx=-\frac{\pi}{2}\int_0^{\pi}\frac{d\cos x}{1+\cos^2 x}$

$$=-\frac{\pi}{2}\arctan(\cos x)\Big|_0^{\pi}=\frac{\pi^2}{4}.$$

例10 设$f(x)=\begin{cases}\dfrac{1}{1+x} & x\ge 0\\[2mm] \dfrac{1}{1+e^x} & x<0\end{cases}$ 求$\int_0^2 f(x-1)dx$.

解 $\because \int_0^2 f(x-1)dx \overset{x-1=t}{=\!=\!=\!=} \int_{-1}^1 f(t)dt = \int_{-1}^0 \dfrac{dx}{1+e^x} + \int_0^1 \dfrac{dx}{1+x}$

$$\overset{e^x=u}{=\!=\!=} \int_{\frac{1}{e}}^1 \frac{du}{(1+u)u} + \ln(1+x)\Big|_0^1 = \ln\frac{u}{1+u}\Bigg|_{\frac{1}{e}}^1 + \ln 2$$

$$= \ln(e+1).$$

例11 设$f(x)$是以T为周期的连续函数，证明$\int_a^{a+T} f(x)dx$的值与a无关.

证 $\because \int_a^{a+T} f(x)dx = \int_a^T f(x)dx + \int_T^{a+T} f(x)dx$

$$\overset{x=T+t}{=\!=\!=\!=\!=} \int_a^T f(x)dx + \int_0^a f(t+T)dt$$

$$= \int_0^a f(t)dt + \int_a^T f(t)dt$$

$$= \int_0^T f(t)dt = \int_0^T f(x)dx.$$

二、定积分的分部积分法

定理 设函数$u(x)$、$v(x)$在区间$[a,b]$上具有连续导数，则有$\int_a^b udv = [uv]_a^b - \int_a^b vdu$.

定积分的分部积分公式

证 $(uv)' = u'v + uv'$,

$$\int_a^b (uv)'dx = [uv]_a^b,$$

$$[uv]_a^b = \int_a^b u'vdx + \int_a^b uv'dx,$$

$$\therefore \int_a^b udv = [uv]_a^b - \int_a^b vdu.$$

定积分的分部积分法习例

例12 计算$\int_0^1 e^{\sqrt{x}}dx$

例13 若$f''(x)$为$[a,b]$上的连续函数,则

$$\int_a^b xf''(x)dx = [bf'(b) - f(b)] - [af'(a) - f(a)]$$

例14 计算$\int_0^{\frac{1}{2}} \arcsin x dx.$

例15 计算$\int_0^{\frac{\pi}{4}} \frac{xdx}{1+\cos 2x}.$

例16 计算$\int_0^1 \frac{\ln(1+x)}{(2+x)^2}dx.$

例17 设$f(x) = \int_1^{x^2} \frac{\sin t}{t}dt$, 求$\int_0^1 xf(x)dx.$

例18 计算$I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx.$

例19 证明$\int_0^x [\int_0^u f(x)dx]du = \int_0^x f(u)(x-u)du.$

例12 计算$\int_0^1 e^{\sqrt{x}}dx$

解 $$\int_0^1 e^{\sqrt{x}}dx \xlongequal[dx=2tdt]{\sqrt{x}=t} \int_0^1 2te^t dt$$

$$= \int_0^1 2t de^t$$

$$= 2te^t\Big|_0^1 - 2\int_0^1 e^t dt$$

$$= 2(te^t - e^t)\Big|_0^1 = 2.$$

例13 若$f''(x)$为$[a,b]$上的连续函数,则

$$\int_a^b xf''(x)dx = [bf'(b)-f(b)]-[af'(a)-f(a)]$$

证 $\because \int_a^b xf''(x)dx = \int_a^b xdf'(x)$

$$= xf'(x)\big|_a^b - \int_a^b f'(x)dx$$

$$= xf'(x)\big|_a^b - f(x)\big|_a^b$$

$$= [bf'(b)-f(b)]-[af'(a)-f(a)].$$

例14 计算 $\int_0^{\frac{1}{2}} \arcsin x dx.$

解 令 $u = \arcsin x, \quad dv = dx,$

则 $du = \dfrac{dx}{\sqrt{1-x^2}}, \quad v = x,$

$$\int_0^{\frac{1}{2}} \arcsin x dx = \left[x \arcsin x\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x dx}{\sqrt{1-x^2}}$$

$$= \frac{1}{2} \cdot \frac{\pi}{6} + \frac{1}{2} \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} d(1-x^2)$$

$$= \frac{\pi}{12} + \left[\sqrt{1-x^2}\right]_0^{\frac{1}{2}} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

例15 计算 $\int_0^{\frac{\pi}{4}} \frac{xdx}{1+\cos 2x}$.

解 $\because 1+\cos 2x = 2\cos^2 x,$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{xdx}{1+\cos 2x} = \int_0^{\frac{\pi}{4}} \frac{xdx}{2\cos^2 x} = \int_0^{\frac{\pi}{4}} \frac{x}{2} d(\tan x)$$

$$= \frac{1}{2}[x \tan x]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} \tan x dx$$

$$= \frac{\pi}{8} + \frac{1}{2}[\ln \cos x]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{\ln 2}{4}.$$

例16 计算 $\int_0^1 \frac{\ln(1+x)}{(2+x)^2}dx.$

解

$$\int_0^1 \frac{\ln(1+x)}{(2+x)^2}dx = -\int_0^1 \ln(1+x)d\frac{1}{2+x}$$

$$= -\left[\frac{\ln(1+x)}{2+x}\right]_0^1 + \int_0^1 \frac{1}{2+x}d\ln(1+x)$$

$$= -\frac{\ln 2}{3} + \int_0^1 \frac{1}{2+x}\cdot\frac{1}{1+x}dx$$

$$\frac{1}{1+x} - \frac{1}{2+x}$$

$$= -\frac{\ln 2}{3} + [\ln(1+x) - \ln(2+x)]_0^1 = \frac{5}{3}\ln 2 - \ln 3.$$

例17 设 $f(x)=\int_1^{x^2}\frac{\sin t}{t}dt$，求 $\int_0^1 xf(x)dx$.

解 因为 $\frac{\sin t}{t}$ 没有初等函数形式的原函数，

无法直接求出 $f(x)$，所以采用分部积分法

$$\int_0^1 xf(x)dx=\frac{1}{2}\int_0^1 f(x)dx^2=\frac{1}{2}\left[x^2f(x)\right]_0^1-\frac{1}{2}\int_0^1 x^2df(x)$$

$$=\frac{1}{2}f(1)-\frac{1}{2}\int_0^1 x^2f'(x)dx \quad \left(f'(x)=\frac{\sin x^2}{x^2}\cdot 2x=\frac{2\sin x^2}{x}\right)$$

$$=0-\frac{1}{2}\int_0^1 2x\sin x^2dx=-\frac{1}{2}\int_0^1 \sin x^2dx^2$$

$$=\frac{1}{2}\left[\cos x^2\right]_0^1=\frac{1}{2}(\cos 1-1).$$

例18 计算$I_n=\int_0^{\frac{\pi}{2}}\sin^n xdx=\int_0^{\frac{\pi}{2}}\cos^n xdx$.

解 $I_n=\int_0^{\frac{\pi}{2}}\sin^{n-1}xd(-\cos x)$

$$=-\cos x\cdot\sin^{n-1}x\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}(n-1)\sin^{n-2}x\cdot\cos^2 xdx$$

$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x(1-\sin^2 x)dx$$

$$=(n-1)\int_0^{\frac{\pi}{2}}(\sin^{n-2}x-\sin^n x)dx$$

$$=(n-1)I_{n-2}-(n-1)I_n \qquad \therefore I_n=\frac{n-1}{n}I_{n-2}$$

$$I_{2m}=\frac{2m-1}{2m}I_{2m-2}=\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdot\frac{2m-5}{2m-4}\cdots\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}I_0$$

$$I_{2m+1}=\frac{2m}{2m+1}I_{2m-1}=\frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdot\frac{2m-4}{2m-3}\cdots\cdots\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}I_1$$

又$I_0=\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{2},\ I_1=\int_0^{\frac{\pi}{2}}\sin x\,dx=1$

$$\therefore I_n=\begin{cases}\dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\cdots\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2} & n\text{为正偶数}\\[2ex] \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\cdots\dfrac{4}{5}\cdot\dfrac{2}{3}\cdot 1 & n\text{为大于1的正奇数}\end{cases}$$

例19 证明$\int_0^x[\int_0^u f(x)dx]du=\int_0^x f(u)(x-u)du.$

证 $\because \frac{d}{du}\int_0^u f(x)dx = f(u)$，由分部积分公式得，

$$\int_0^x[\int_0^u f(x)dx]du=[u\int_0^u f(x)dx]_0^x-\int_0^x uf(u)du$$

$$=x\int_0^x f(x)dx-\int_0^x uf(u)du$$

$$=x\int_0^x f(u)du-\int_0^x uf(u)du$$

$$=\int_0^x (x-u)f(u)du.$$

内容小结

基本积分法 { 换元积分法 / 分部积分法 }

- 换元积分法
- 分部积分法

换元必换限
配元不换限
边积边代限